Trautman-Bondi energy and its universality

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Gravitational waves

1916 – Einstein describes gravitational waves. He uses linearized version of general relativity theory and derives his famous ``quadrupole formula''.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Minkowski flat metric

perturbation

20' – 30' – Einstein begins to have doubts about validity of such an appraoch:

$$\begin{aligned} g \;=\; & \boxed{-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \\ & \boxed{\begin{aligned} &- \cos(x-t)\,(2+\cos(x-t))\,\mathrm{d}t^2 \\ &+2\cos(x-t)\,(1+\cos(x-t))\,\mathrm{d}t\mathrm{d}x - \cos^2(x-t)\mathrm{d}x^2 \end{aligned}} \end{aligned}$$

Minkowski flat metric

perturbation

Gravitational waves

$$
\begin{aligned}
g \;=\; & -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
& - \cos(x-t)\,(2+\cos(x-t))\,\mathrm{d}t^2 \\
& + 2\cos(x-t)\,(1+\cos(x-t))\,\mathrm{d}t\mathrm{d}x - \cos^2(x-t)\mathrm{d}x^2
\end{aligned}
$$

Choose a new variable: $\tilde{t} := t + \sin(t-x)$

$$
g \;=\; -\mathrm{d}\tilde{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2
$$

Main difficulty: how to decouple ``gauge freedom'' from the ``true degrees of freedom''. (Einstein writes several papers.)

1937 – Einstein ``proves'' that gravitational waves do not exist!

``Red light'': serious people do not believe in gravitational waves (including Leopold Infeld – father of the polish theoretical physics.)

Gravitational waves

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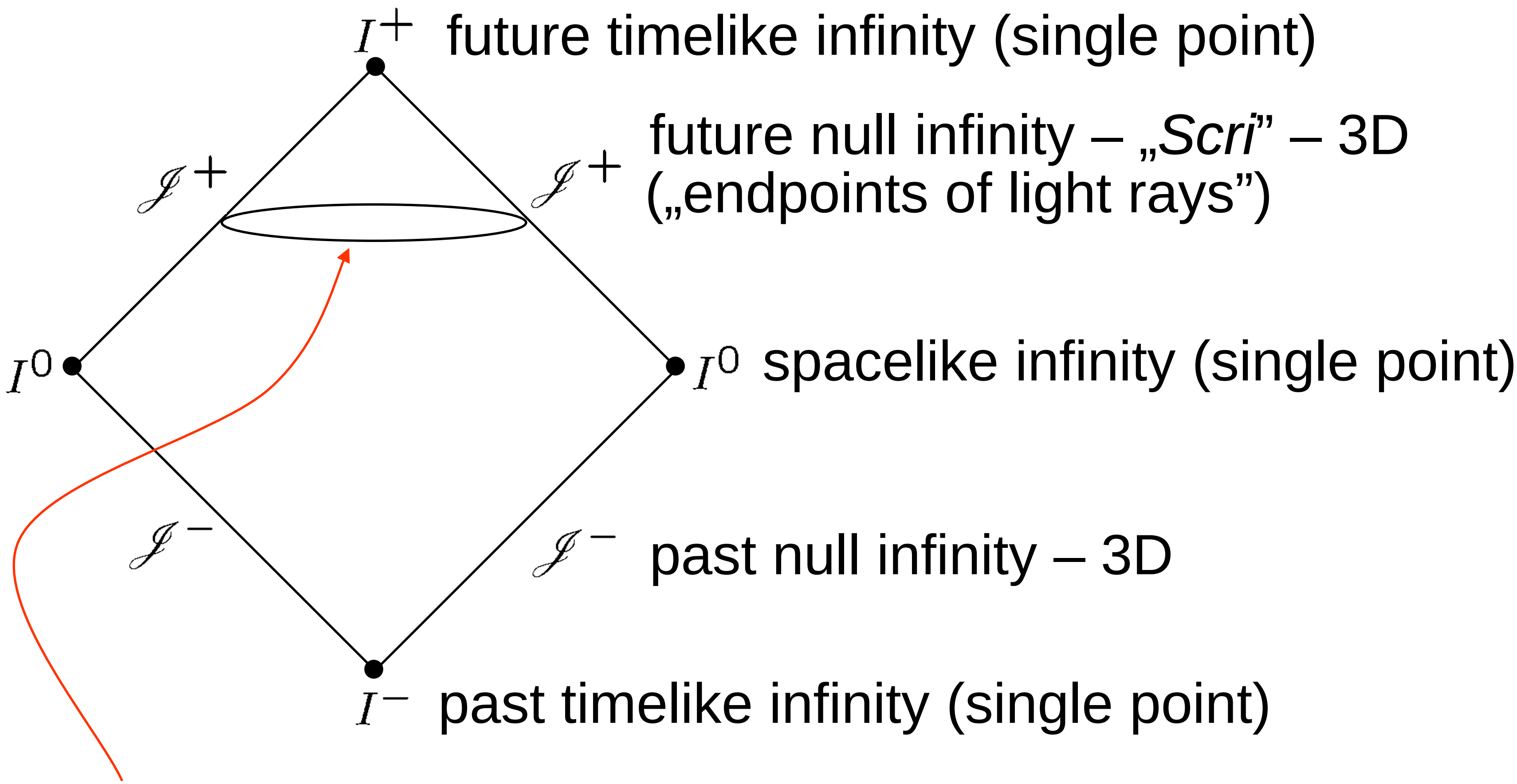
1958–Andrzej Trautman Ph.D thesis: formulation of conceptual framework and mathematical tools which are necessary to describe gravitational radiation.

1) Radiation is not a local phenomenon.
2) It is localized ``at infinity''.
3) Boundary conditions are fundamental:
   generalization of the ``Sommerfeld radiation conditon'' in special relativity („asymptotic flattness" in General Relativity).

1964 – Roger Penrose: conformal treatemant of infinity.

Compactified space-time



To each 2-dimensional slice we may assigne energy which has not been radiated yet („Trautman-Bondi energy").

Na skraju!

Compactified space-time

Idea of the conformal compactification:

$$
\begin{aligned}
g &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\sigma^2
\end{aligned}
$$

line element
on a unit sphere
$\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\varphi^2$

$$
u := t - r\,, \qquad v := t + r\,.
$$

$$
g = -\mathrm{d}u\mathrm{d}v + \frac{1}{4}(v-u)^2 \mathrm{d}\sigma^2
$$

$$
u := \tan U\,, \qquad v := \tan V\,, \quad -\pi/2 < U, V < \pi/2\,.
$$

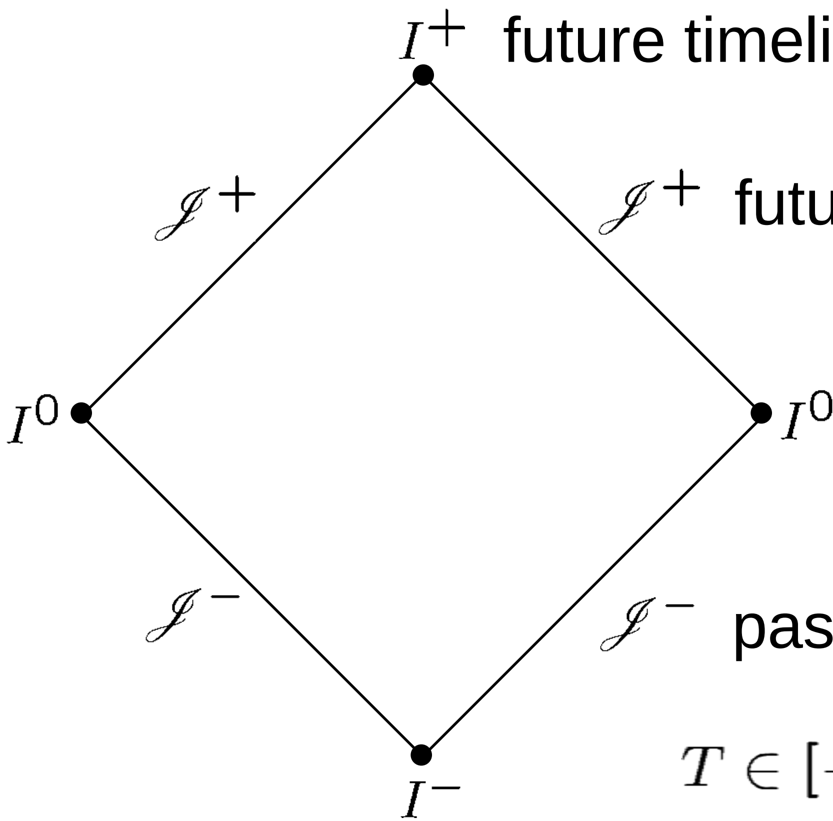
$$
g = \frac{1}{4\cos^2 U \cos^2 V}\left[-\mathrm{d}U\mathrm{d}V + \frac{1}{4}\sin^2(V-U)\mathrm{d}\sigma^2\right]
$$

$$
T = U + V\,, \qquad R = V - U\,.
$$

$$
g = \frac{1}{4\cos^2 U \cos^2 V}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \frac{1}{4}\sin^2 R \mathrm{d}\sigma^2\right]
$$

$$
T \in [-\pi, \pi]\,, \qquad R \in [0, \pi - |t|]\,.
$$

Compactified space-time



I^+ future timelike infinity $\quad T = +\pi\,, \qquad R = 0$

$\mathscr{I}^+$ future null infinity $\quad T + R = \pi$

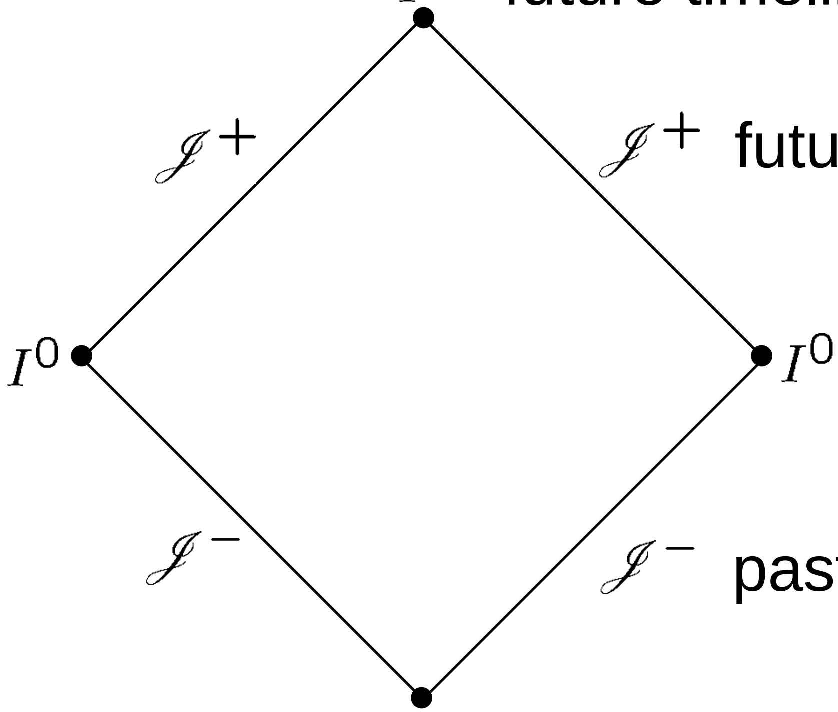
I^0 spacelike infinity $\quad \begin{aligned} T &= 0 \\ R &= \pi \end{aligned}$

$\mathscr{I}^-$ past null infinity $\quad R - T = \pi$

$$T \in [-\pi, \pi/]\,, \qquad R \in [0, \pi - |t|]\,.$$

$$g \;=\; \frac{1}{4\cos^2 U \cos^2 V}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \frac{1}{4}\sin^2 R \mathrm{d}\sigma^2\right]$$

Compactified space-time



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I^0 spacelike infinity $T = 0$, $R = \pi$

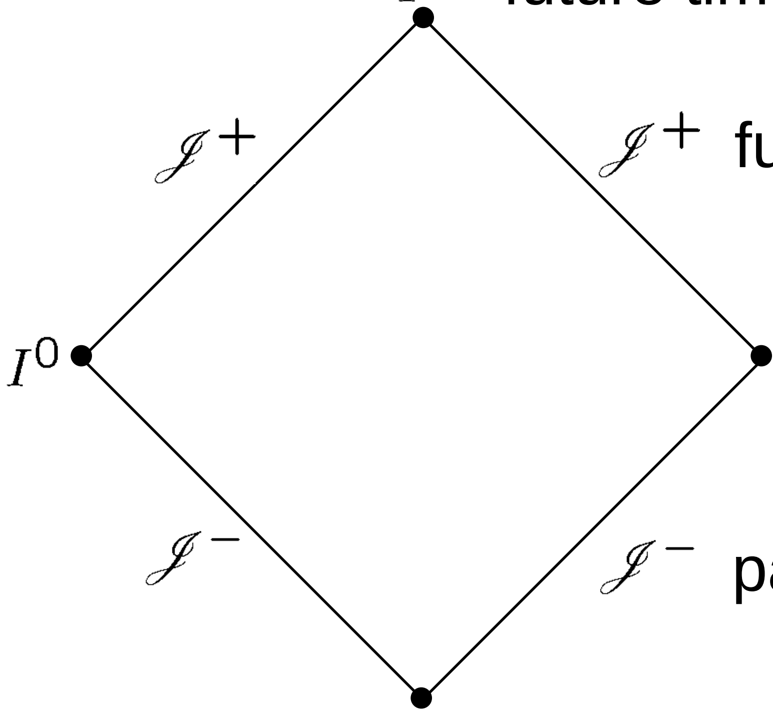
$\mathscr{I}^-$ past null infinity $R - T = \pi$

I^- past timelike infinity $T = -\pi\,, \quad R = 0$

conformal factor

$$g = \frac{1}{4\cos^2 U \cos^2 V}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \frac{1}{4}\sin^2 R \mathrm{d}\sigma^2\right]$$

Compactified space-time



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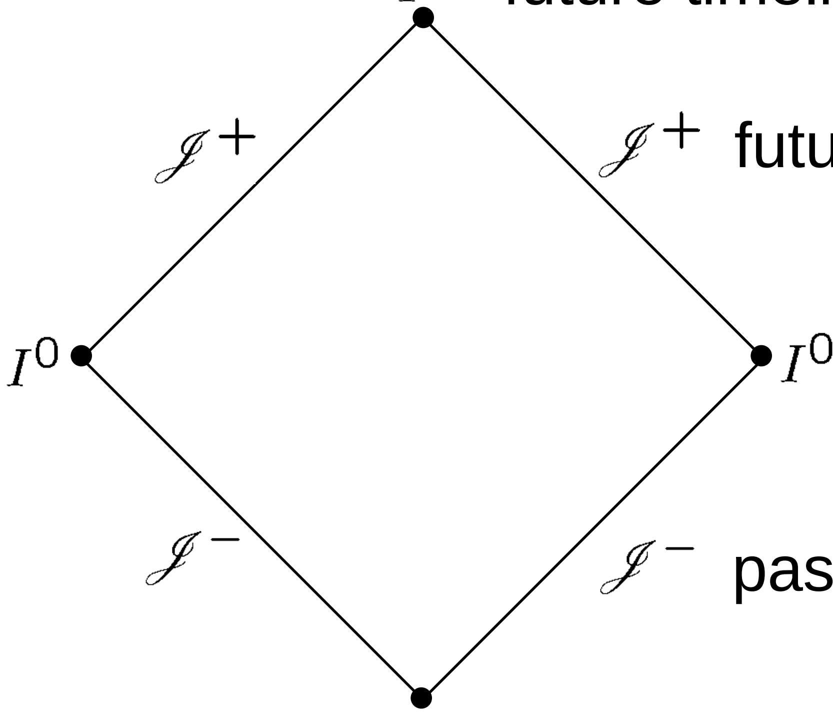
I^- past timelike infinity $\quad T = -\pi\,, \quad R = 0$

conformal factor

$$g \;=\; \Omega^{-2}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \frac{1}{4}\sin^2 R\mathrm{d}\sigma^2\right]$$

Compactified space-time

I^+ future timelike infinity $\quad T = +\pi\,, \quad R = 0$



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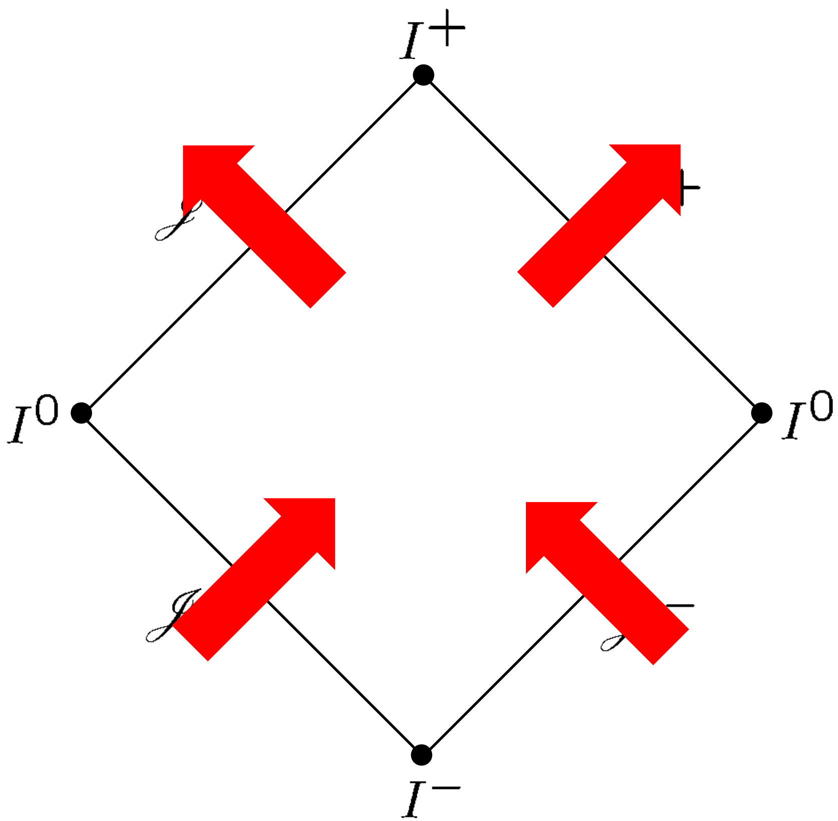
conformal factor

$$g = \Omega^{-2}\tilde{g}$$

$\boxed{\text{fictitious metric}}$

Compactified space-time



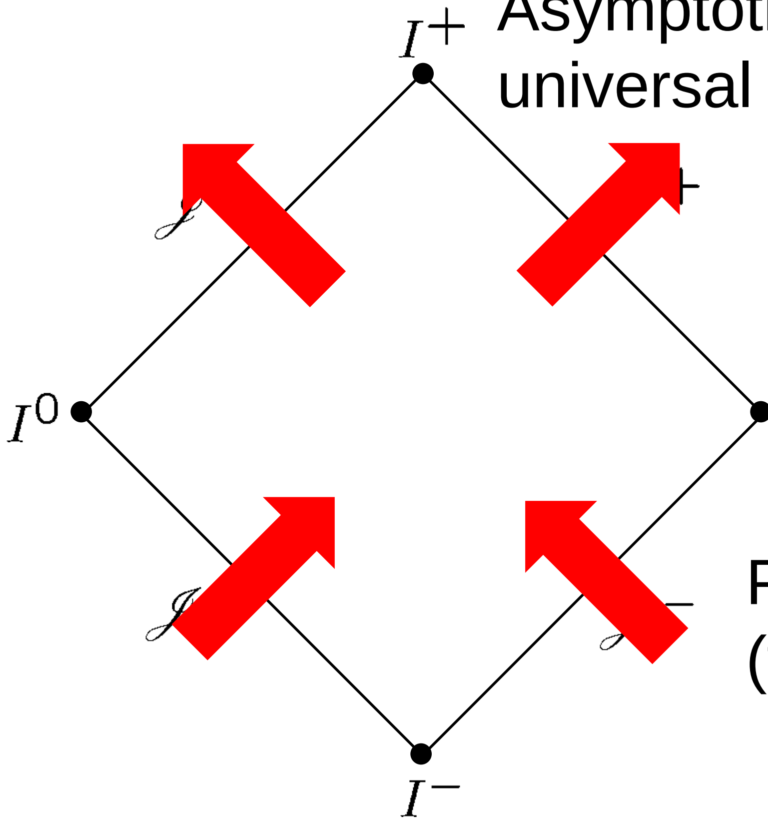
Outgoing radiation

Incoming radiation

$$g = \Omega^{-2}\tilde{g}$$

fictitious metric

Universal propertis of radiation



Asymptotic behaviour of radiation is universal - not specific to general relativity.

In asymptotic regime: dynamic of gravitational field = two decoupled wave equations.

Penrose' metric description (fictitious!) of the „Scri" is obsolete.

Consider wave equation on Minkowski space:

$$\Box\phi = \left(-\mathrm{d}^2/\mathrm{d}t^2 + \Delta\right)\phi = 0$$

Wave equation

$$\Box\phi = \left(-\mathrm{d}^2/\mathrm{d}t^2 + \Delta\right)\phi = 0$$

Every solution of wave equation is uniquely determined by initial (Cauchy) data:

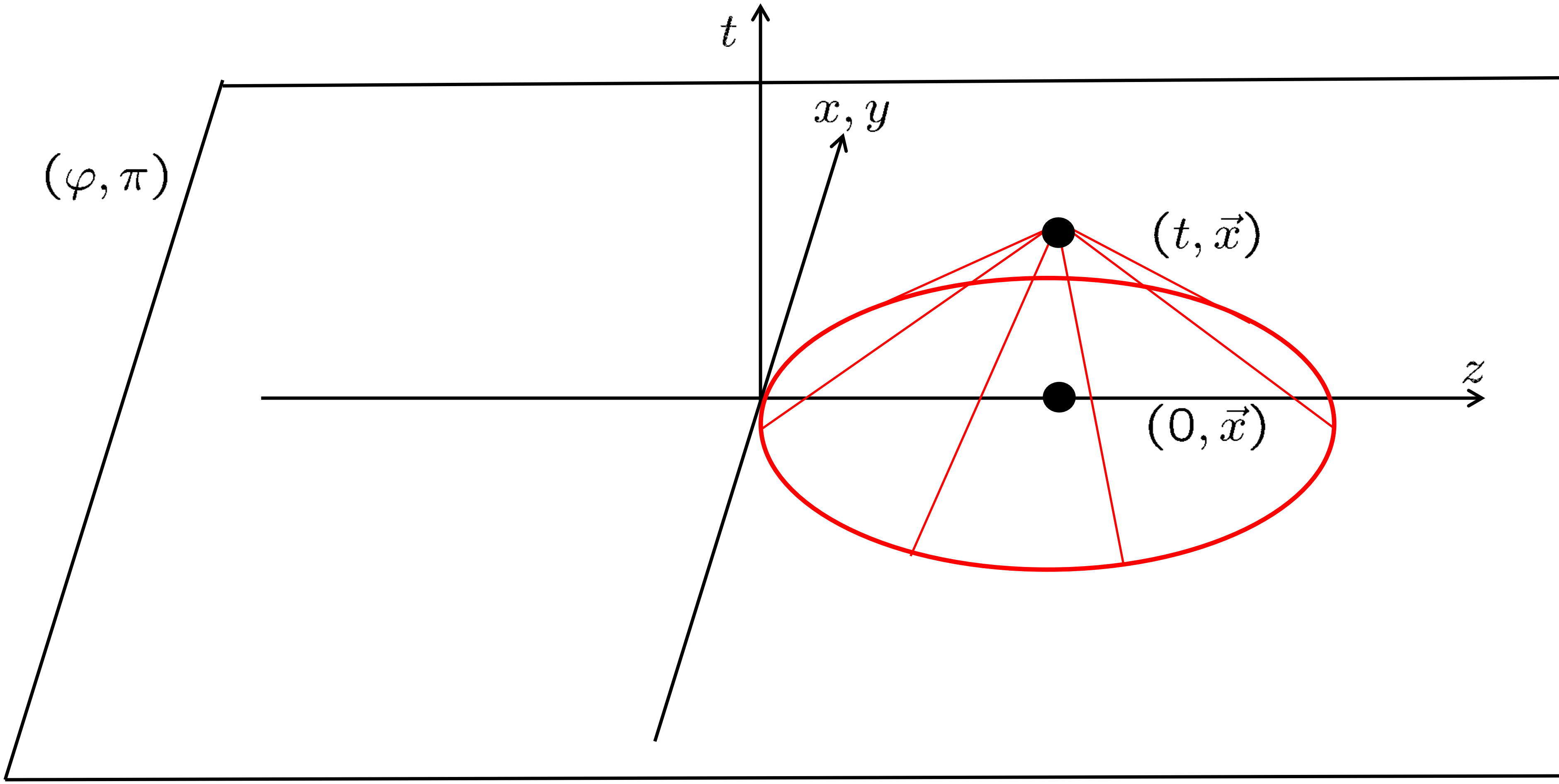
$$\phi(0,\vec{x}) = \varphi(\vec{x})\,, \qquad \frac{\partial}{\partial t}\phi(0,\vec{x}) = \pi(\vec{x})\ .$$

$$\phi(t,\vec{x}) := \frac{\partial}{\partial t}\left(t\cdot \underbrace{\overline{\varphi(S(\vec{x},|t|))}}\right) + t\cdot \underbrace{\overline{\pi(S(\vec{x},|t|))}}\ .$$

Mean value of the function over the sphere centered at $\vec{x}$ whose radius is: $|t|$

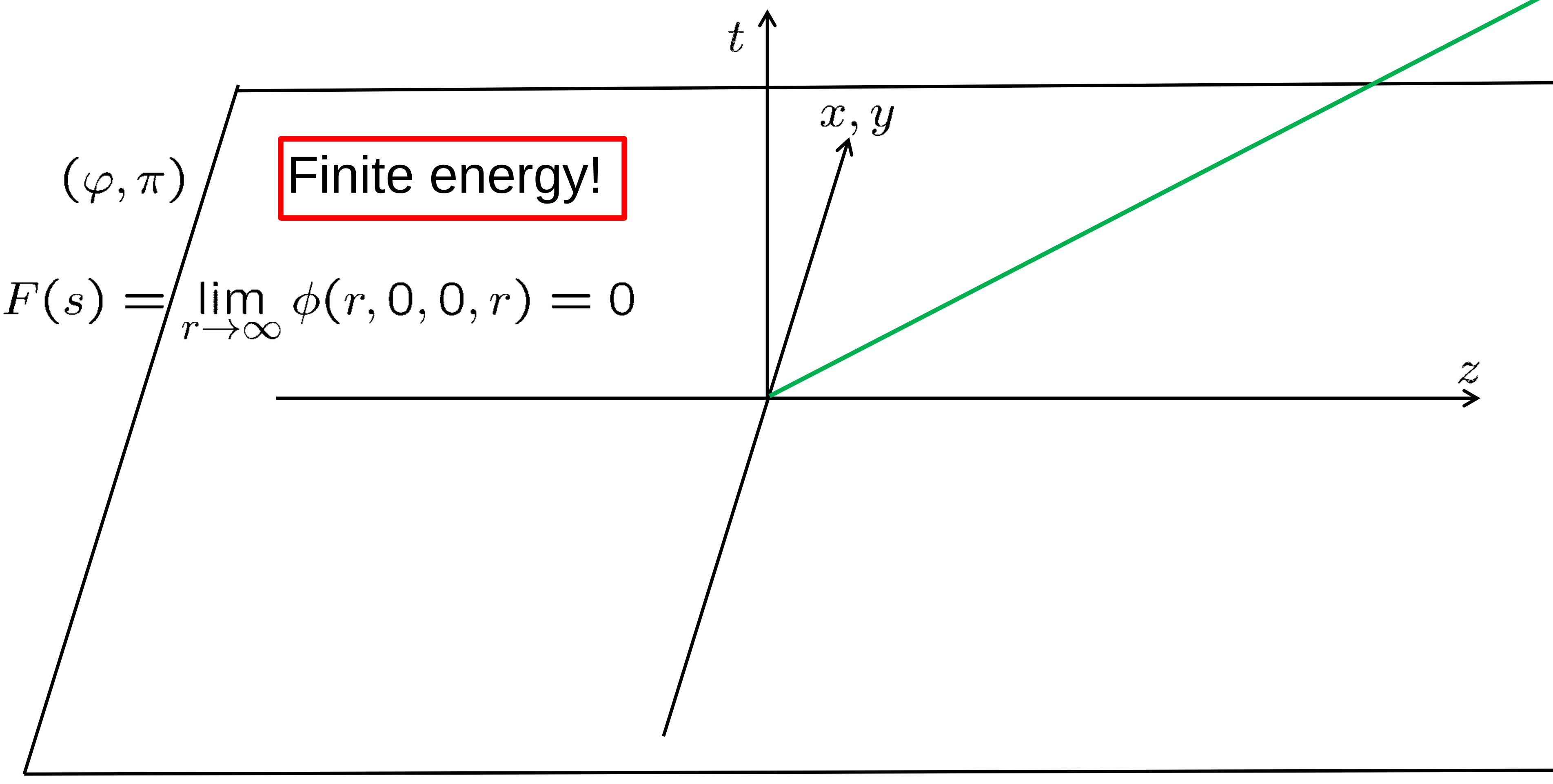
„Huygens formula"

Wave equation



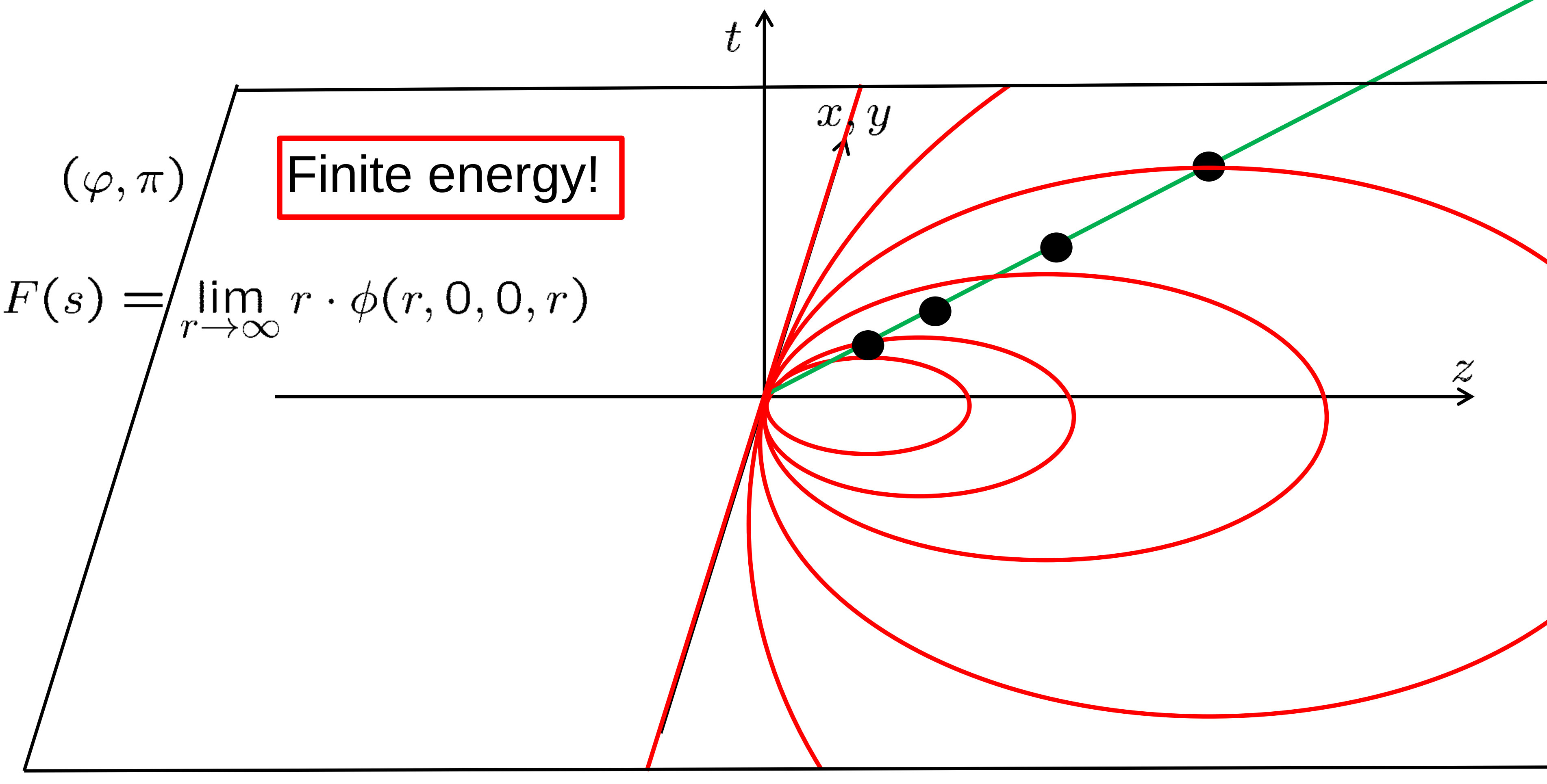
$$\phi(t,\vec{x}) := \frac{\partial}{\partial t}\left(t \cdot \overline{\varphi(S(\vec{x},|t|))}\right) + t \cdot \overline{\pi(S(\vec{x},|t|))}\ .$$

Wave equation



Point on the scri – „endpoint” of a light ray.

Wave equation



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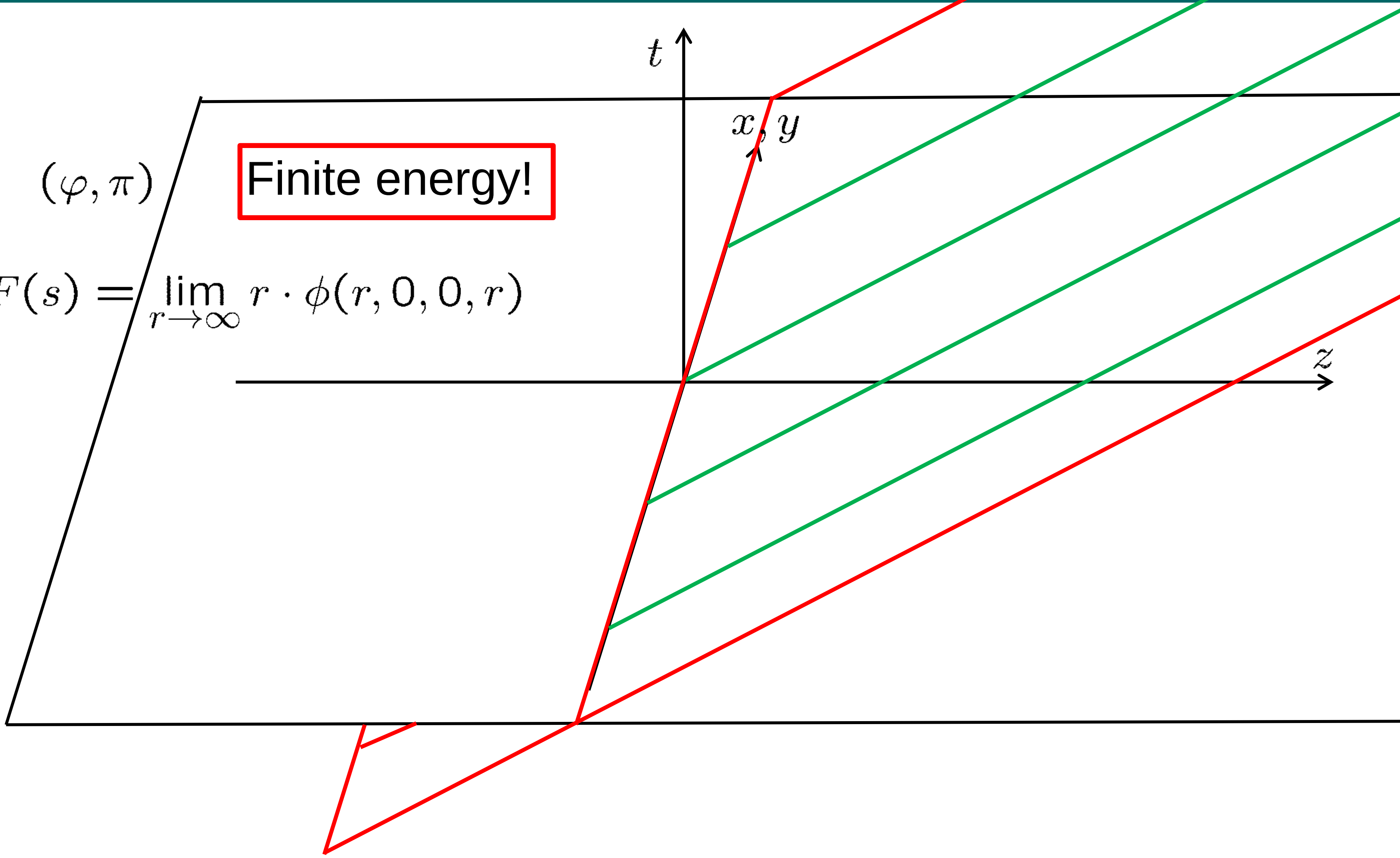
Wave equation



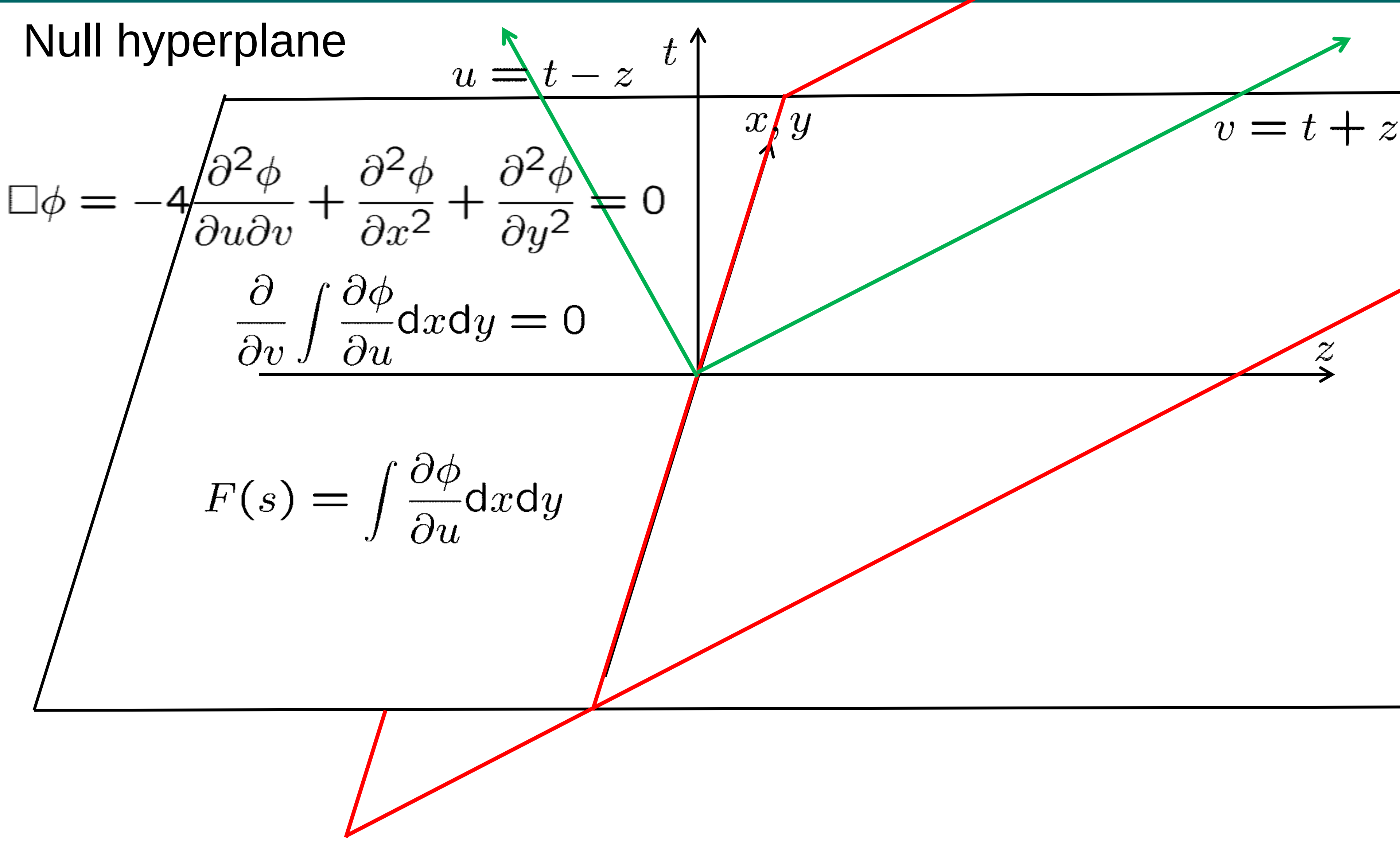
(φ, π)

Finite energy!

$$F(s) = \lim_{r \to \infty} r \cdot \phi(r, 0, 0, r)$$

Radiation data on Scri

Null hyperplane



$$\Box\phi = -4\frac{\partial^2\phi}{\partial u\partial v} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\frac{\partial}{\partial v}\int\frac{\partial\phi}{\partial u}\mathrm{d}x\mathrm{d}y = 0$$

$$F(s) = \int\frac{\partial\phi}{\partial u}\mathrm{d}x\mathrm{d}y$$

Radiation data on Scri

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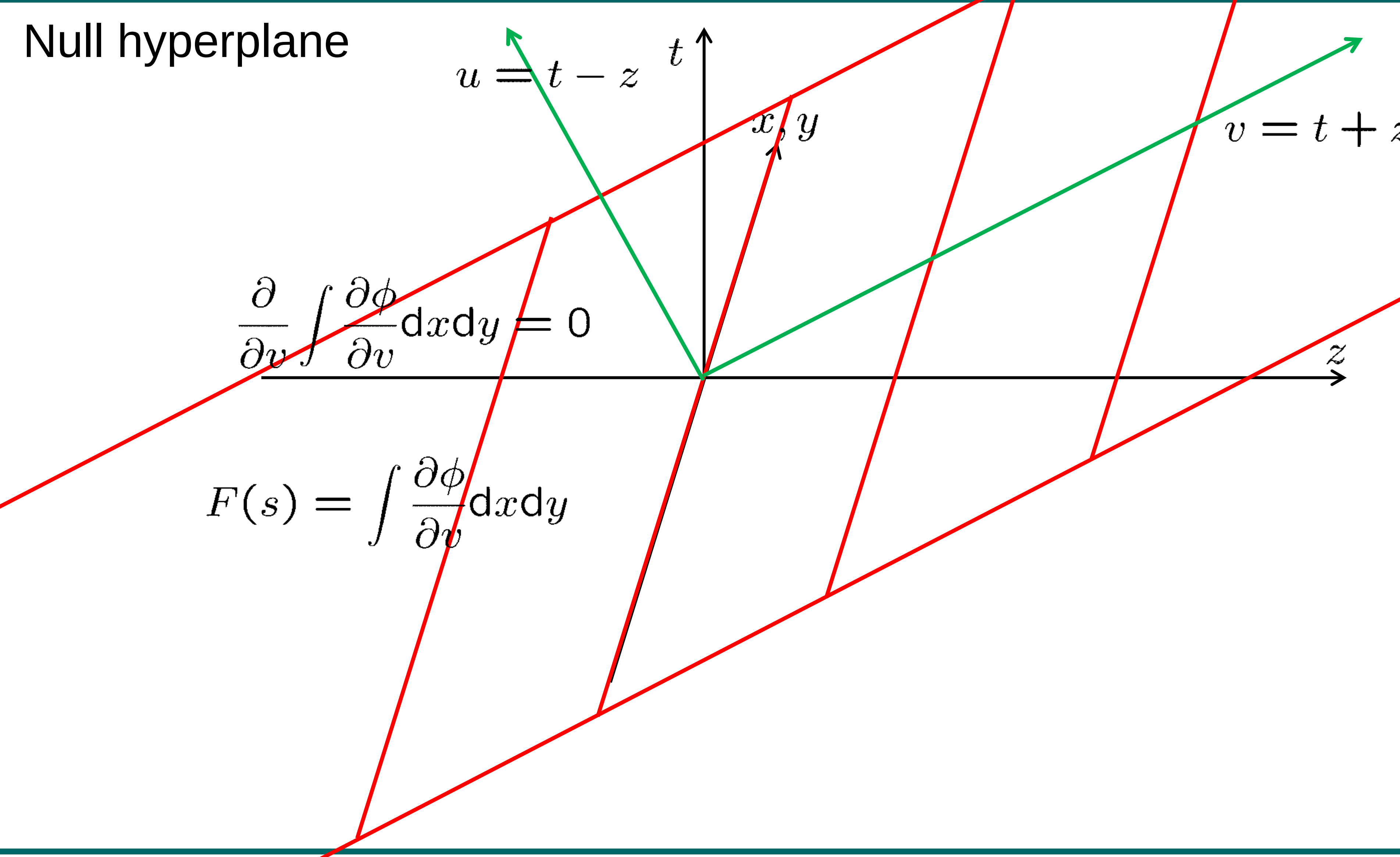
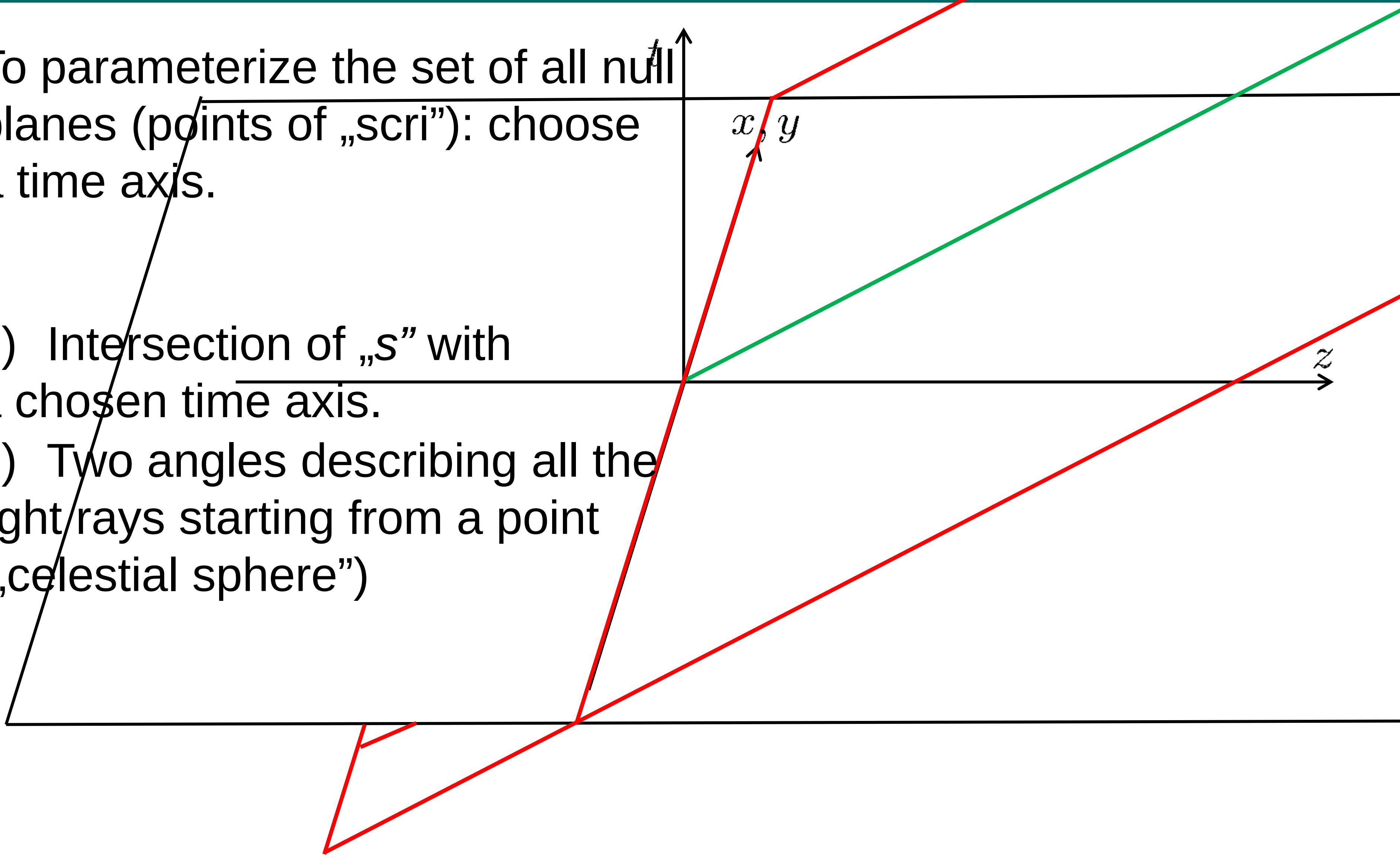


$$\frac{\partial}{\partial v}\int \frac{\partial \phi}{\partial v}\mathrm{d}x\mathrm{d}y = 0$$

$$F(s) = \int \frac{\partial \phi}{\partial v}\mathrm{d}x\mathrm{d}y$$

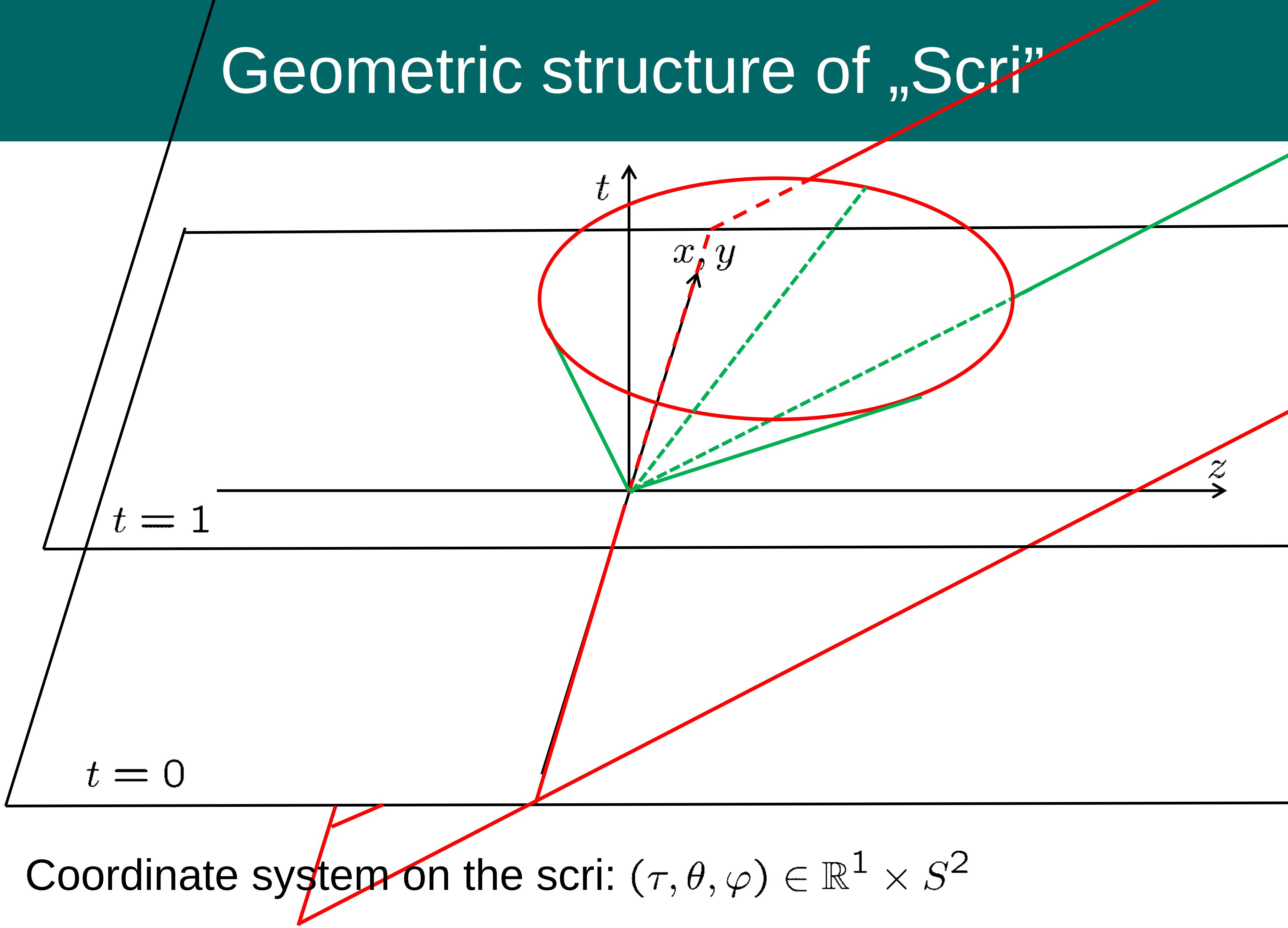
Geometric structure of „Scri"

To parameterize the set of all null planes (points of „scri"): choose a time axis.



1) Intersection of „*s*" with a chosen time axis.
2) Two angles describing all the light rays starting from a point („celestial sphere")

Geometric structure of „Scri”



Coordinate system on the scri: $(\tau, \theta, \varphi) \in \mathbb{R}^1 \times S^2$

Initial data *vs.* radiation data

Theorem: Transformation between initial (Cauchy) data (π, φ) and the radiation data F is a symplectomorphism (canonical transfrormation) of the two canonical structures:

$$\Omega = \delta p \wedge \delta q$$

$$\Omega_{Cauchy} = \int \delta\pi(\vec{x}) \wedge \delta\varphi(\vec{x}) d^3x$$

$$\Omega_{Cauchy}\left((\pi, \varphi), (\pi', \varphi')\right) = \int \left(\pi(\vec{x})\varphi'(\vec{x}) - \pi'(\vec{x})\varphi(\vec{x})\right) d^3x$$

$$\Omega_{Radiation} = \int \delta\frac{\partial F}{\partial \tau} \wedge \delta F d\tau d\mu$$

$$\Omega_{Radiation}(F, G) = \int \frac{\partial F}{\partial \tau} G d\tau d\mu$$

$d\mu = \sin\theta d\theta d\varphi$

measure on S^2

The two structures can be treated as different representations of the same phase space describing possible field configurations.

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$$\{\pi(\vec{x}), \varphi(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})$$

$$d\mu = \sin\theta d\theta d\varphi$$

measure on S^2

$$\Omega_{Radiation} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F d\tau d\mu$$

$$\{F(\tau, \theta, \varphi), F(\tilde{\tau}, \tilde{\theta}, \tilde{\varphi})\} = \delta(\theta - \tilde{\theta})\delta(\varphi - \tilde{\varphi})\delta'(\tau - \tilde{\tau})$$

The two structures can be treated as different representations of the same phase space describing possible field configurations.

Lorentz invariance

$$\Omega_{Radiation} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F d\tau d\mu$$

Symplectic structure in space of radiation data is not, a priori, Lorentz invariant: depends upon the choice of the time axis!

What happens if we change the time axis?

1) Time parameter changes.

2) Change of the volume element $d\mu$ on the celestial sphere.

3) Change of the value of $F(s) = \lim_{r \to \infty} r \cdot \phi(r, 0, 0, r)$

But, the miracle occurs:

$$\mathcal{F} := F \cdot \sqrt{d\mu}$$

remains unchaned!!!

$$\boxed{\Omega_{Radiation} = \int \delta \frac{\partial \mathcal{F}}{\partial \tau} \wedge \delta \mathcal{F} d\tau}$$

Field energy

Time evolution of the field is generated by the Hamiltonian (field energy).

In „Cauchy picture”:

$$\Omega_{Cauchy} = \int \delta\pi \wedge \delta\varphi$$

$$\begin{aligned} \dot{\varphi} &= \pi &&= \frac{\delta\mathcal{H}}{\delta\pi} \\ \dot{\pi} &= \Delta\varphi &&= -\frac{\delta\mathcal{H}}{\delta\varphi} \end{aligned}$$

$$\mathcal{H} = \frac{1}{2}\int \left(\pi^2 + (\nabla\varphi)^2\right) d^3x \quad \geq 0$$

$$-\varphi\Delta\varphi$$

In „radiation picture”:

$$\Omega_{Radiation} = \int \delta\underbrace{\frac{\partial\mathcal{F}}{\partial\tau}}_{\mathcal{P}} \wedge \delta\mathcal{F} d\tau$$

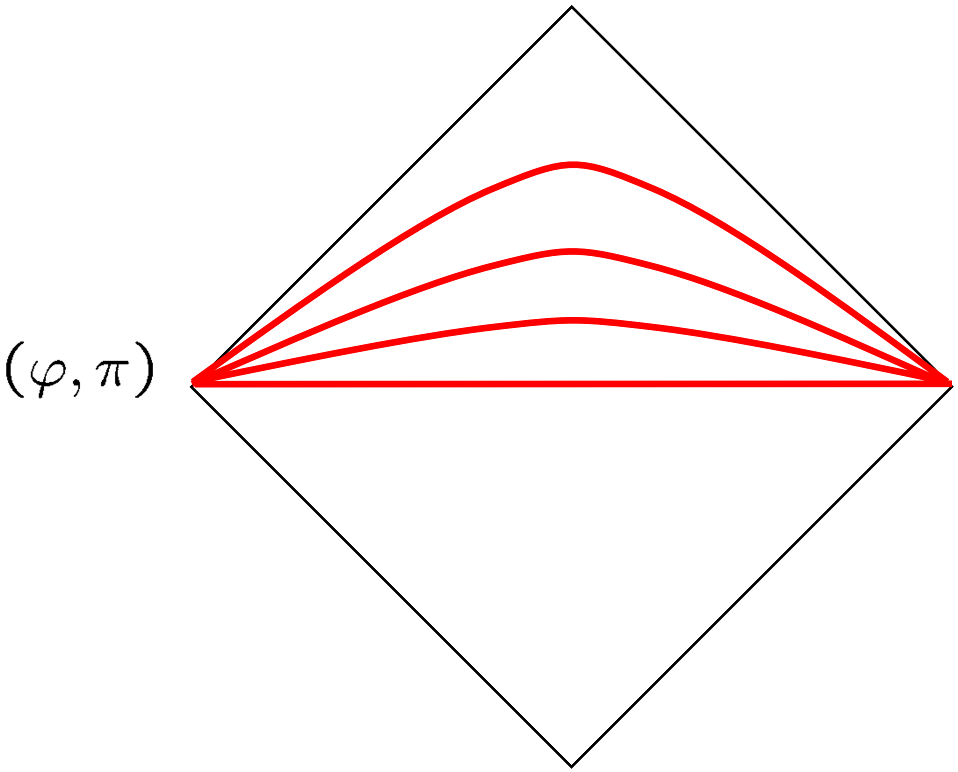
time evolution = translation in parameter τ

$$\begin{aligned} \dot{\mathcal{F}} &= \partial_\tau\mathcal{F} &&= \frac{\delta\mathcal{H}}{\delta\mathcal{P}} \\ \dot{\mathcal{P}} &= \partial_\tau\mathcal{P} &&= -\frac{\delta\mathcal{H}}{\delta\mathcal{F}} \end{aligned}$$

$$\begin{aligned} \mathcal{H} &= \int (\mathcal{P}\partial_\tau\mathcal{F})\, d\tau \\ &= \int (\partial_\tau\mathcal{F})^2\, d\tau \quad \geq 0 \end{aligned}$$

(like momentum in the conventional Cauchy formulation)

Time evolution



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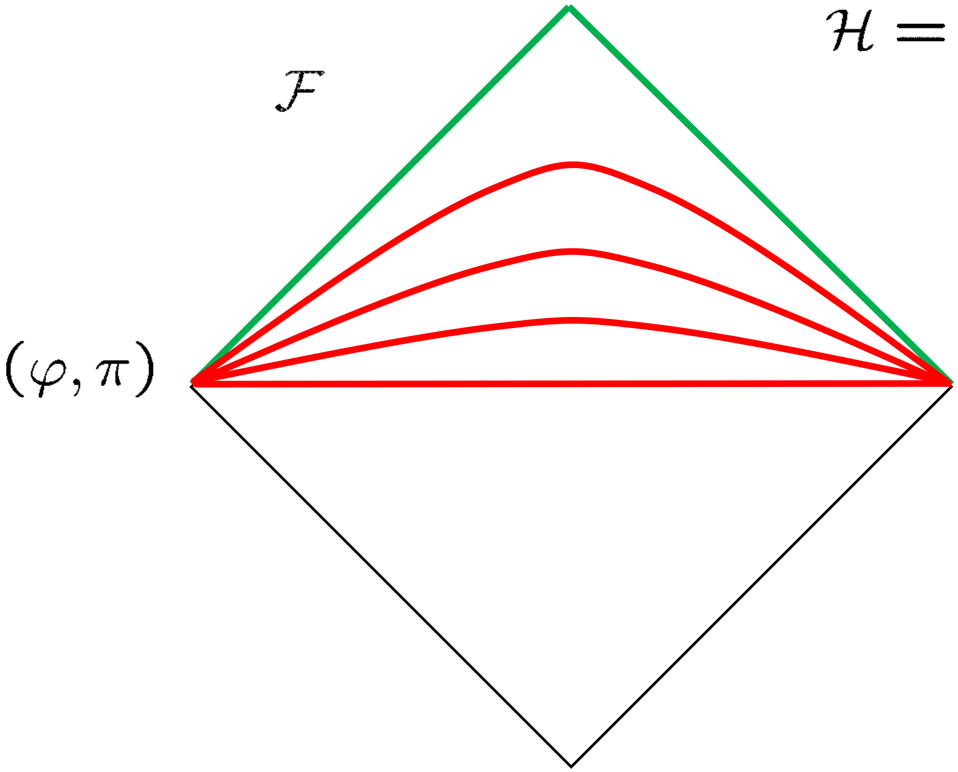
Time evolution



$$\mathcal{H} = \int (\partial_\tau \mathcal{F})^2 \, d\tau$$

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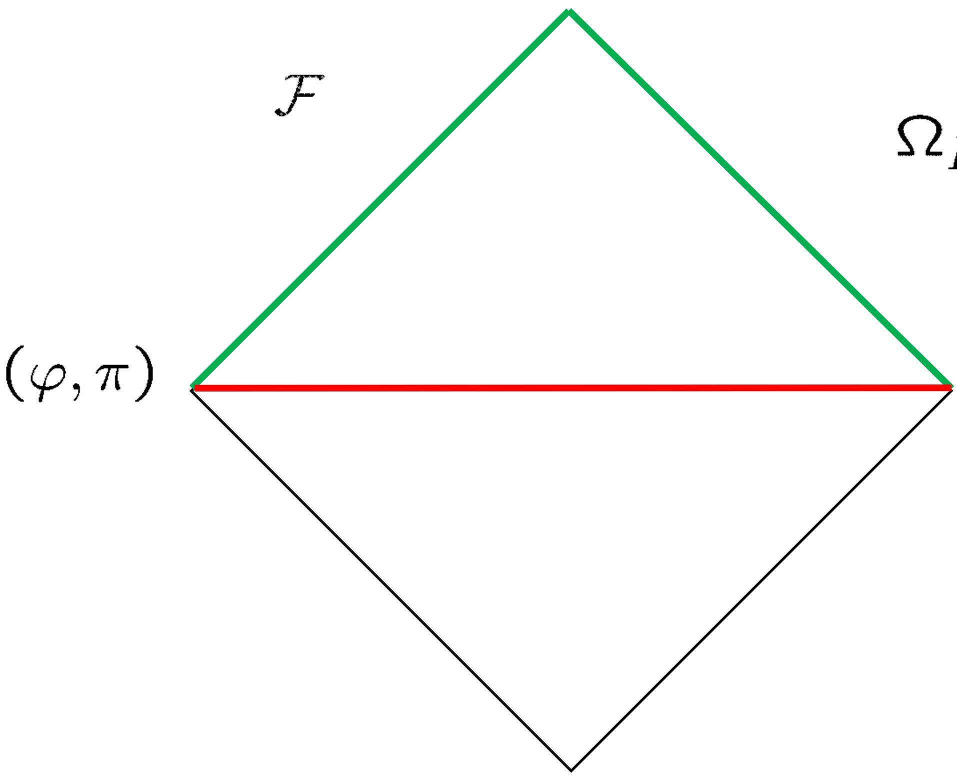
Time evolution



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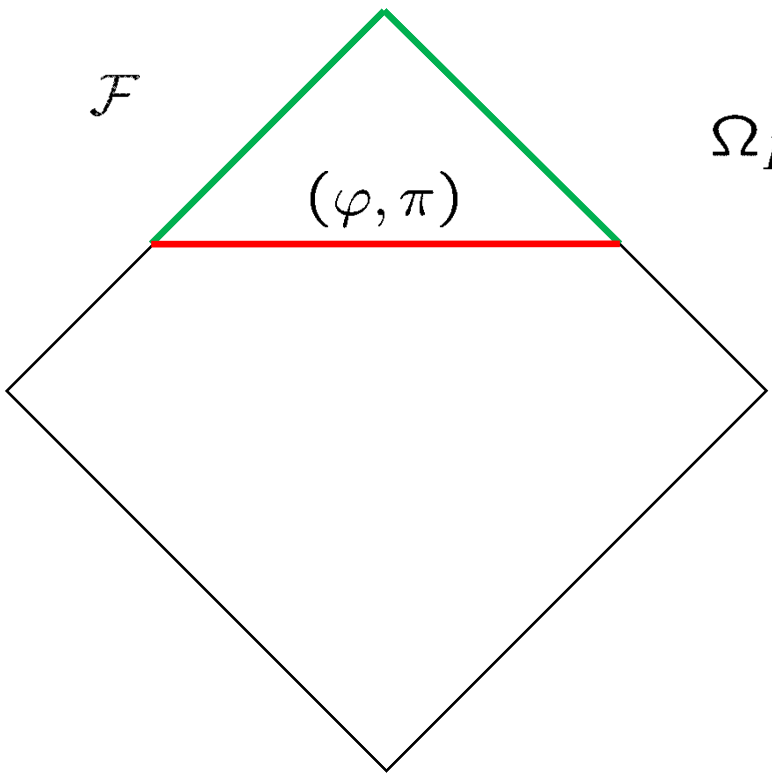
Quantization???

Cauchy problem on a hyperboloid



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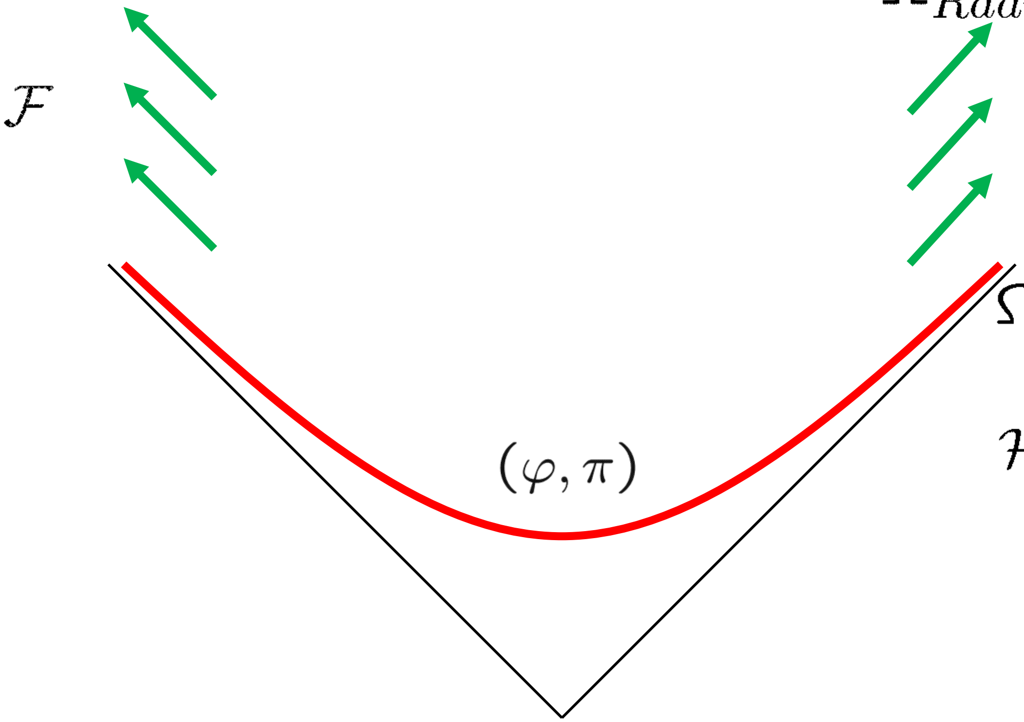

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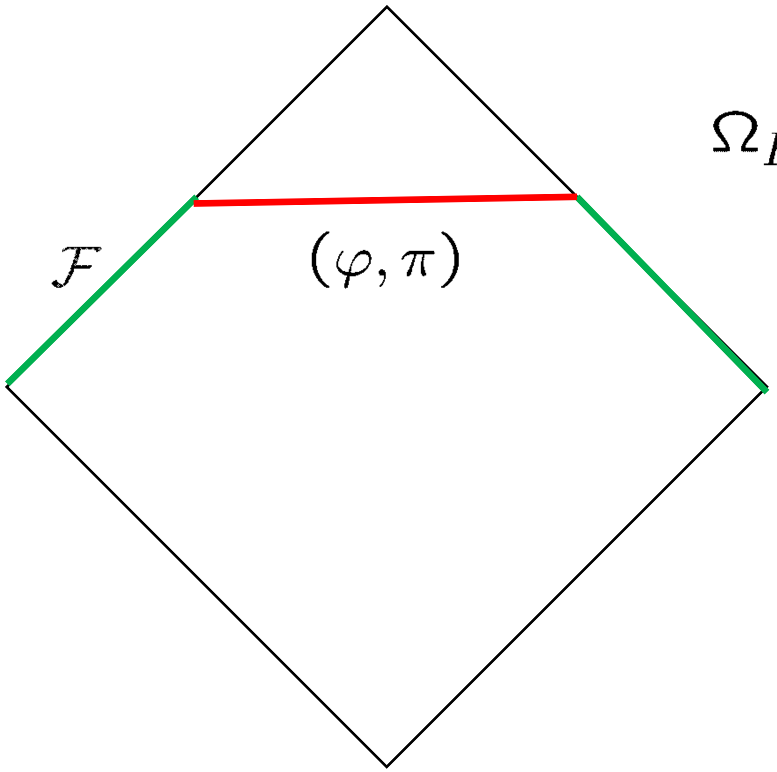
Mixed: „Cauchy–Radiation” picture



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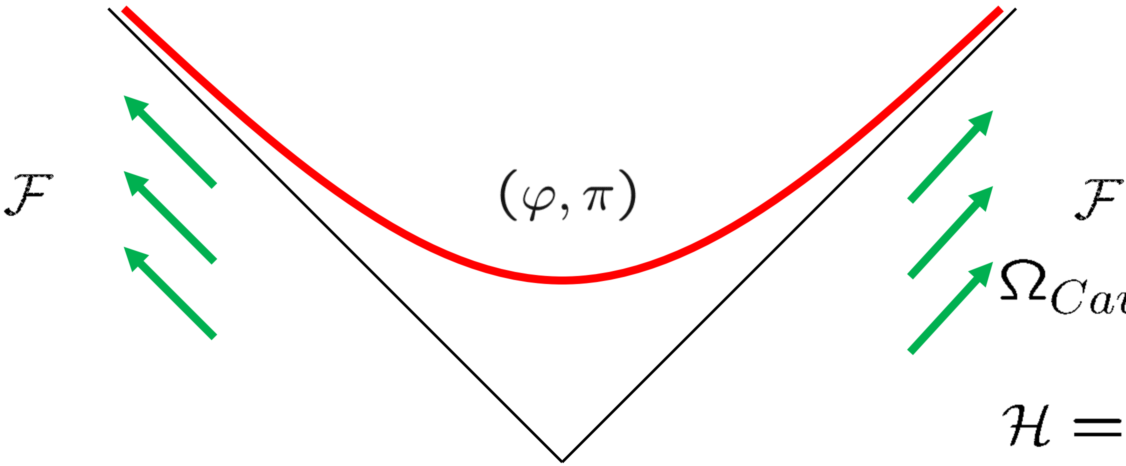

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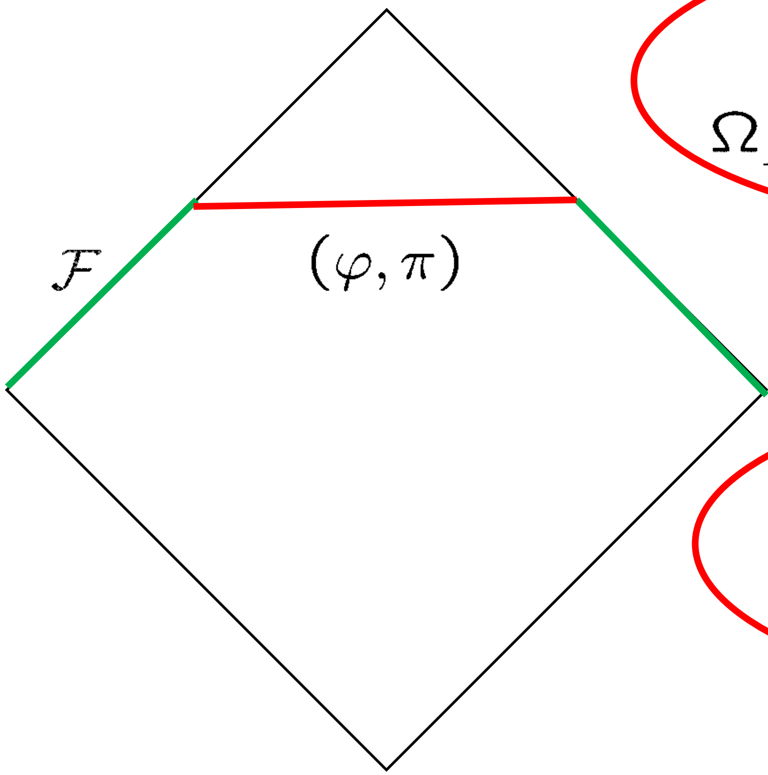

$$\mathcal{H}^- = \int (\partial_\tau \mathcal{F})^2 \, d\tau$$

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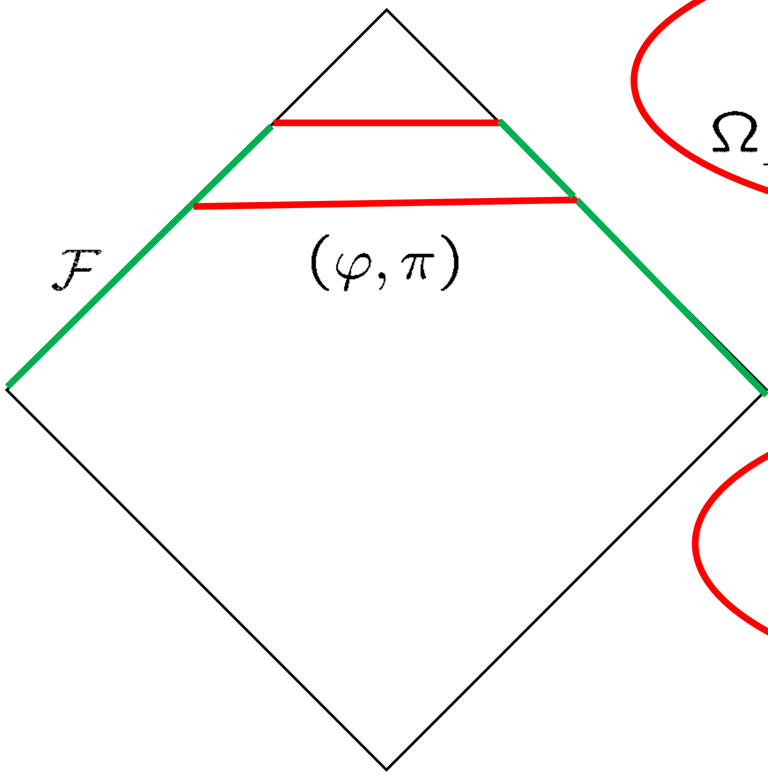
$$\Omega_{Cauchy} = \int \delta\pi \wedge \delta\varphi$$

$$\mathcal{H}^+ = \frac{1}{2} \int \left(\pi^2 + (\nabla\varphi)^2 \right) d^3x$$

$$\mathcal{H}^{total} = \mathcal{H}^- + \mathcal{H}^+$$

Mixed: „Cauchy–Radiation" picture



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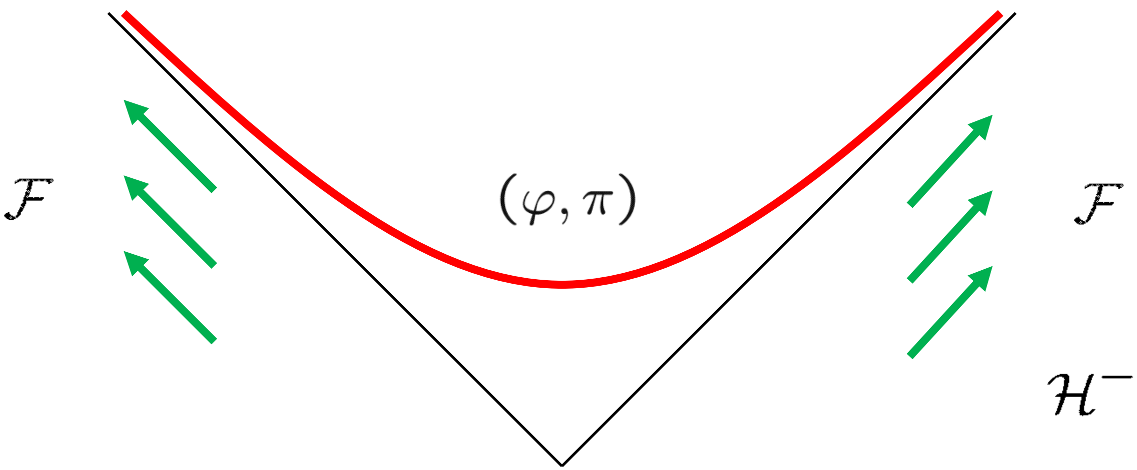
$$\mathcal{H}^{total} = \mathcal{H}^- + \mathcal{H}^+ \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{total} = 0 \qquad \mathcal{H}^-(t) = \int_{-\infty}^{t} (\partial_\tau \mathcal{F})^2 \, d\tau$$

$\mathcal{H}^-$ is strictly increasing, whence, $\mathcal{H}^+$ is strictly decreasing

Mixed: „Cauchy–Radiation” picture

$$\mathcal{H}^{total} = \mathcal{H}^- + \mathcal{H}^+ = \text{const.}$$

$\mathcal{H}^+ \searrow 0$ Trautman-Bondi energy



$\mathcal{H}^- \nearrow \mathcal{H}$

$$\mathcal{H}^- = \int (\partial_\tau \mathcal{F})^2 \, d\tau$$

$$\mathcal{H}^-(t) = \int_{\tau<t} (\partial_\tau \mathcal{F})^2 \, d\tau$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^+(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^-(t) = -\,(\partial_t \mathcal{F})^2$$

Technicalities

Technical features which are specific to General Relativity:

No null hyper-**planes** (but asymptotically…)

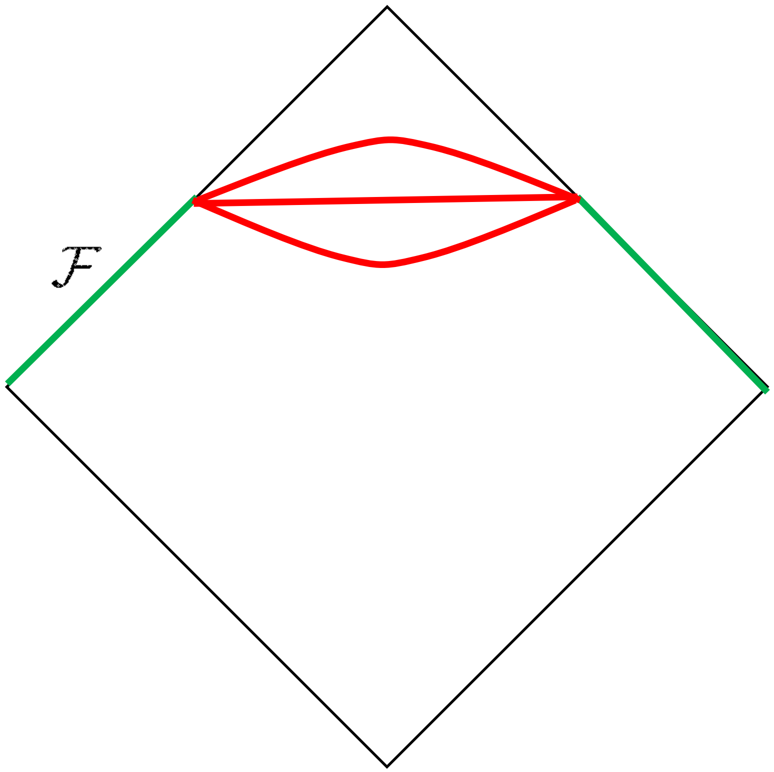
Meaningful physical quantities never defined by volume integrals but only by surface integrals.

Hence: shape of the hyperbolid does not matter!

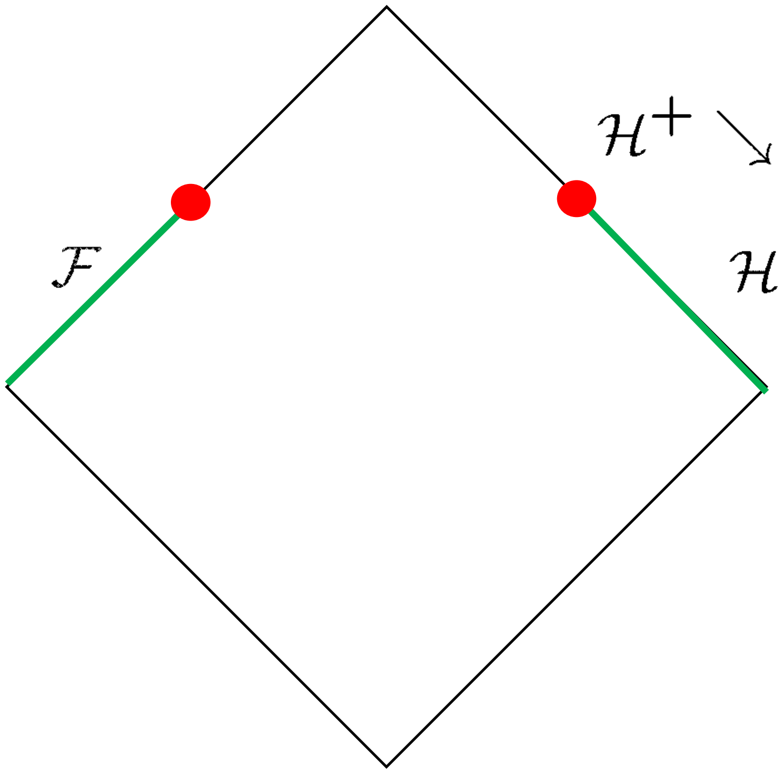
What matters is its intersection with the Scri.

Cauchy energy density is given by a complete divergence of a „ superpotential” (i.e. „Freud”, „Landau-Lifshitz” or any other) and, whence, Trautman-Bondi energy can be calculated as a surface integral over any 2D surface on Scri.

Mixed: „Cauchy–Radiation” picture



Mixed: „Cauchy–Radiation” picture



$\mathcal{H}^{+} \searrow 0$ Trautman-Bondi energy

$\mathcal{H}^{-} \nearrow \mathcal{H}$ already radiated energy

$$\mathcal{H}^{total} = \mathcal{H}^{-} + \mathcal{H}^{+} = \text{const.}$$